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FINITE GROUPS WITH ABSOLUTELY $\mathfrak{F}$-SUBNORMAL MAXIMAL SUBGROUPS[1]

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A subgroup M of a group G is an n-maximal subgroups of G if there is a subgroup chain $M = M_n \leq M_{n-1} \leq \ldots \leq M_1 \leq M_0 = G$ such that M_{i+1} is a maximal subgroup of M_i. We establish a criterion for a group with absolutely $\mathfrak{F}$-subnormal n-maximal subgroups to belong to a subgroup-closed saturated formation $\mathfrak{F}$ containing all nilpotent groups.

Keywords: finite group, maximal subgroup, subnormal subgroup.

И. Л. Сохор. Конечные группы с абсолютно $\mathfrak{F}$-субнормальными максимальными подгруппами.

Подгруппа M группы G является n-максимальной подгруппой в группе G, если существует цепочка подгрупп $M = M_n \leq M_{n-1} \leq \ldots \leq M_1 \leq M_0 = G$ такая, что M_{i+1} — максимальная подгруппа в M_i. Для группы с абсолютно $\mathfrak{F}$-субнормальными n-максимальными подгруппами установлен критерий принадлежности наследственной насыщенной формации $\mathfrak{F}$, содержащей все нильпотентные группы.

Ключевые слова: конечная группа, максимальная подгруппа, субнормальная подгруппа.

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Introduction

All groups in this paper are finite.

The structure of a group depends in large measure on the properties of its maximal subgroups, in particular, the manner of embedding a maximal subgroup into the group. Remind that a proper subgroup M of a group G is called maximal in G if whenever $M \leq H \leq G$ we have $M = H$ or $H = G$. A subgroup M is an n-maximal subgroup of a group G if there is a subgroup chain

$$M = M_n \leq M_{n-1} \leq \ldots \leq M_1 \leq M_0 = G$$

such that M_{i+1} is a maximal subgroup of M_i for every i.

Huppert [1] proved that a group with all 2-maximal subgroups normal is supersolvable and a group with all 3-maximal subgroups normal is a solvable group of rank at most 2. Janko [2] described groups in which 4-maximal subgroups are normal. Mann [3] investigated groups with all n-maximal subgroups subnormal for arbitrary n.

The concept of formational subnormality is a generalization of the concept of subnormality. Let $\mathfrak{F}$ be a formation. A subgroup H of a group G is $\mathfrak{F}$-subnormal in G if $H = G$ or there is a subgroup chain

$$H = H_0 \lessdot H_1 \lessdot \ldots \lessdot H_n = G$$

such that $H_i/{H_{i-1}}_{H_i} \in \mathfrak{F}$ for every i (or, equivalently, $H_i^{\mathfrak{F}} \leq {H_{i-1}}_{H_i}$). Here we write $H \lessdot G$ if H is a maximal subgroup of G and we denote by $H_G = \bigcap_{g \in G} H^g$ the core of H in G. A subgroup H

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of a group G is called absolutely $\mathfrak{F}$-subnormal in G if any subgroup L containing H is $\mathfrak{F}$-subnormal in G [4]. It is clear that any $\mathfrak{F}$-subnormal maximal subgroup is absolutely $\mathfrak{F}$-subnormal.

If $\mathfrak{F}$ is a formation containing all nilpotent groups, then every subnormal subgroup of a solvable group is $\mathfrak{F}$-subnormal, in particular, $\mathfrak{N}$-subnormal (see Lemma 3). Here $\mathfrak{N}$ is the formation of all nilpotent groups. In [5], the structure of a group with $\mathfrak{F}$-subnormal 2-maximal subgroups was established for a lattice formation $\mathfrak{F}$. Groups with $\mathfrak{F}$-subnormal n-maximal subgroups with additional restrictions on the number of prime divisors of the group order were studied in [6; 7].

Thus, it is natural to study a group in which n-maximal subgroups are absolutely $\mathfrak{F}$-subnormal. In this paper, we establish a criterion for a group with absolutely $\mathfrak{F}$-subnormal n-maximal subgroups to belong to a subgroup-closed saturated formation $\mathfrak{F}$ containing all nilpotent groups.

1. Preliminaries

Let $G \neq 1$ be a group. Then there is an unrefinable subgroup chain

$$1 = M_k \lessdot M_{k-1} \lessdot \ldots \lessdot M_1 \lessdot M_0 = G.$$

The number of subgroups in this chain is called its length. The length $l(G)$ of a group G is the maximal length of an unrefinable chain. The depth $\lambda(G)$ of G is the minimal length of an unrefinable chain.

Lemma 1. *Let G be a group. If $\lambda(G) = 1$, then G is a group of prime order.*

Proof. Since $\lambda(G) = 1$, then $1 \lessdot G$, and so G is a group of prime order. Lemma is proved.

Lemma 2. *Let G be a group. If $\lambda(G) = 2$, then $|\pi(G)| \leq 2$ and one of the following statements holds.*

(1) *G is a group of prime power order.*

(2) *$G = C_p \times C_q$ for some primes p and q with $p \neq q$.*

(3) *G is a non-nilpotent group and every proper subgroup of G is a group of prime power order.*

Proof. Since $\lambda(G) = 2$, there is a subgroup chain

$$1 \lessdot M \lessdot G.$$

Therefore, $|M| = q$ is prime and G is solvable by [8, IV.7.4]. A maximal subgroup of a solvable group has a prime power index. Therefore, we consider the following cases.

(1) $q \in \pi(|G : M|)$. Then $|G| = q^\alpha$ and G is a group of prime power order.

(2) $q \notin \pi(|G : M|)$. Then $|G| = p^\alpha q$ for a prime $p \in \pi(G)$ and $p \neq q$. If M is normal in G, then $|G : M| = p$, $|G| = pq$ and $G = C_q \rtimes C_p$. If, in addition, $p < q$, then $G = C_p \times C_q$. Let M be not normal in G. Then $M = N_G(M)$ is a Sylow q-subgroup of G, and according to [8, IV.2.6], there is a q'-Hall subgroup N such that $G = N \rtimes M$. Since M is a maximal subgroup of G, we conclude that N is a minimal normal subgroup of G. Consequently, N is an elementary abelian p-group because G is solvable. Suppose that there is a proper subgroup H of G such that $|\pi(H)| = 2$. It follows that $M \leq H < G$ and $M = H$, but $|\pi(M)| = 1$, a contradiction. Thus, every proper subgroup of G has a prime power order.

Lemma is proved.

Note that groups with $\lambda(G) = 3$ or $\lambda(G) = 4$ were described in [9].

Lemma 3. *Let $\mathfrak{F}$ be a formation containing all nilpotent groups. Every subnormal subgroup of a solvable group G is $\mathfrak{F}$-subnormal in G.*

Proof. Let H be a subnormal subgroup of G. There is a composition series

$$1 \lessdot \ldots \lessdot H = H_0 \lessdot H_1 \lessdot \ldots \lessdot H_n = G.$$

Since G is solvable, we have $|H_{i+1} : H_i|$ is prime for every i. Therefore, $H_{i+1}/H_i \in \mathfrak{N} \subseteq \mathfrak{F}$ for every i and H is $\mathfrak{F}$-subnormal in G.

Lemma is proved.

Lemma 4. *Let $\mathfrak{F}$ be a subgroup-closed formation. If $G \in \mathfrak{F}$, then every subgroup of G is absolutely $\mathfrak{F}$-subnormal.*

Proof. Let H be a subgroup of a group $G \in \mathfrak{F}$. There is subgroup chain

$$H = H_0 \lessdot H_1 \lessdot \ldots \lessdot H_n = G.$$

Since $\mathfrak{F}$ is a subgroup-closed formation and $G \in \mathfrak{F}$, we deduce $H_i \in \mathfrak{F}$ and $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{F}$ for every i. Consequently, H is $\mathfrak{F}$-subnormal in G. Thus, every subgroup of G is $\mathfrak{F}$-subnormal in G. Therefore every subgroup containing H is $\mathfrak{F}$-subnormal in G and H is absolutely $\mathfrak{F}$-subnormal in G.

Lemma is proved.

2. The main result

Lemma 5. *Let $\mathfrak{F}$ be a formation and let G be a simple group. If G contains an $\mathfrak{F}$-subnormal subgroup, then $G \in \mathfrak{F}$.*

Proof. Let H be an $\mathfrak{F}$-subnormal subgroup of a simple group G. In that case there is a subgroup chain

$$H = H_0 \lessdot H_1 \lessdot \ldots \lessdot H_{n-1} = M \lessdot H_n = G$$

such that $H_i/H_{i-1 H_i} \in \mathfrak{F}$ for every i. In particular, $G/M_G \cong G \in \mathfrak{F}$ in view of G is a simple group.

Lemma is proved.

Theorem 1. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups and let G be a group with all n-maximal subgroups absolutely $\mathfrak{F}$-subnormal. The following statements hold.*

(1) *If $n \le 2$, then $G \in \mathfrak{F}$.*

(2) *If $3 \le n \le 4$, then either $G \in \mathfrak{F}$ or G is a solvable group of chief length is no more than $n-1$.*

(3) *If $n \ge 5$, then either $G \in \mathfrak{F}$ or G is a solvable group of chief length is no more than $n-1$ or G is an unsolvable group with $3 \le \lambda(G) \le n-1$.*

Proof. Let $n = 1$. Since every maximal subgroup of G is absolutely $\mathfrak{F}$-subnormal in G, every maximal subgroup of G is $\mathfrak{F}$-subnormal in G. Therefore $G \in \mathfrak{F}$ by [10, Lemma 4].

Let $n = 2$ and let M be a maximal subgroup of G. If $M = 1$, then $|G|$ is prime and $G \in \mathfrak{N} \subseteq \mathfrak{F}$. Assume that $M \neq 1$. In that case there is a subgroup chain

$$K \lessdot M \lessdot G.$$

Since K is a 2-maximal subgroup of G, we deduce that K is absolutely $\mathfrak{F}$-subnormal in G by the choice of G. Hence M is $\mathfrak{F}$-subnormal in G and $G \in \mathfrak{F}$ according to [10, Lemma 4].

Let $n > 2$ and $G \notin \mathfrak{F}$. If $\lambda(G) = 1$, then by Lemma 1, $G \in \mathfrak{F}$, a contradiction. Hence $\lambda(G) \ge 2$. Suppose that $\lambda(G) \ge n$. In that case for every maximal subgroup M of G, there is a subgroup chain

$$M_n \lessdot \ldots \lessdot M_1 = M \lessdot G_0 = G.$$

Since M_n is an n-maximal subgroup of G, we get M_n is absolutely $\mathfrak{F}$-subnormal in G by the choice of G. Hence M is $\mathfrak{F}$-subnormal in G and $G \in \mathfrak{F}$ by [10, Lemma 4], a contradiction. Consequently, $\lambda(G) \leq n-1$.

Let $n = 3$. Then $\lambda(G) = 2$ by the above. In view of Lemma 2, we obtain that G is solvable. Consequently, the length of a chief series of G is equal to 2 since the depth of a solvable group is equal to the length of its chief series by [11, Theorem 2].

Let $n = 4$. Then $2 \leq \lambda(G) \leq 3$ by the above. If $\lambda(G) = 2$, then according to Lemma 2, G is solvable, and the chief length of G is equal to 2 by [11, Theorem 2]. Let $\lambda(G) = 3$. If G is solvable, then the chief length of G is equal to 3 by [11, Theorem 2]. Let G be unsolvable. If $l(G) = \lambda(G) = 3$, then G is supersolvable in view of [12], a contradiction. Hence $l(G) > 3$ and G contains a 4-maximal subgroup that is absolutely $\mathfrak{F}$-subnormal in G. By [9, Theorem 1], G is simple. Consequently, $G \in \mathfrak{F}$ by Lemma 5, a contradiction.

Let $n \geq 5$. By the above, $2 \leq \lambda(G) \leq n-1$. If G is solvable, then the chief length of G is no more than $n-1$ by [11, Theorem 2]. Assume that G is unsolvable. If $\lambda(G) = 2$, then G is solvable according to Lemma 2, a contradiction. Therefore we have $3 \leq \lambda(G) \leq n-1$.

Theorem is proved.

Corollary 1. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups. If G is an unsolvable group with all n-maximal subgroups absolutely $\mathfrak{F}$-subnormal ($n \leq 4$), then $G \in \mathfrak{F}$.*

Proof. If $n \leq 2$, then $G \in \mathfrak{F}$ by Theorem 1. Let $2 < n \leq 4$. Assume that $G \notin \mathfrak{F}$. According to Theorem 1, G is a solvable group, a contradiction.

Corollary is proved.

Corollary 2. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups. Every 3-maximal subgroup of a group G is absolutely $\mathfrak{F}$-subnormal if and only if either $G \in \mathfrak{F}$ or every primary cyclic subgroup of G is absolutely $\mathfrak{F}$-subnormal or self-normalizing.*

Proof. Assume that every 3-maximal subgroup of a group G is absolutely $\mathfrak{F}$-subnormal in G. By Theorem 1, either $G \in \mathfrak{F}$, or G is a solvable group of chief length 2. Let $G \notin \mathfrak{F}$. Since G is solvable, we deduce that $\lambda(G) = 2$ by [11, Theorem 2]. Hence G is a non-nilpotent group in which every proper subgroup is of prime power order in view of Lemma 2. Consequently, every primary cyclic subgroup of G is absolutely $\mathfrak{F}$-subnormal or self-normalizing by [10, Theorem 2].

Conversely, if $G \in \mathfrak{F}$, then by Lemma 4, every subgroup of G is absolutely $\mathfrak{F}$-subnormal in G. Assume that $G \notin \mathfrak{F}$ and every primary cyclic subgroup of G is absolutely $\mathfrak{F}$-subnormal or self-normalizing. According to [10, Theorem 2], G is a non-nilpotent group in which every proper subgroup is of prime power order. By [13], we get $G = P \rtimes \langle x \rangle$, where P is an elementary abelian Sylow p-group for a prime $p \in \pi(G)$, $\langle x \rangle$ is a non-normal Sylow subgroup of order q for a prime $q \in \pi(G)$, $p \neq q$ and $\langle x \rangle$ acts irreducibly on P. In that case, every 3-maximal subgroup K is contained in P. Hence K and every subgroup H of G containing K are subnormal in G. Since G is solvable, K as well as every subgroup containing K is $\mathfrak{F}$-subnormal in G by Lemma 3, and so, K is absolutely $\mathfrak{F}$-subnormal in G.

Corollary is proved.

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