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SOLUTION OF A PARABOLIC HAMILTON–JACOBI TYPE EQUATION DETERMINED BY A SIMPLE BOUNDARY SINGULARITY

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For a parabolic Hamilton–Jacobi type equation $S_t + 2^{-1}(S_x)^2 + V(x,\varepsilon) = S_{xx}$, a special asymptotic solution with a prescribed asymptotic expansion of the potential function is constructed. Since this asymptotic expansion is chosen for simplicity in the form of a series in natural powers of the small parameter ε, the asymptotic solution of the equation is presented in the form of a series from perturbation theory in integer powers of ε: $S(x,t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n S_n(x,t)$. The leading approximation of the solution is expressed in terms of an exponential integral as follows:

$$S_0(x,t) = -2\ln \int\limits_0^{+\infty} \exp\left(-\sigma^3 + t\sigma^2 + x\sigma\right) d\sigma,$$

where the versal deformation of the germ of the simple boundary singularity B_3 serves as the phase. The asymptotic behavior of this integral in the space variable at infinity is studied by the Laplace method. On the basis of an integral recurrence formula with a homogeneous initial condition for the remaining coefficients $S_n(x,t)$, an existence theorem is proved. Exponential estimates of these coefficients are also established; they provide the convergence of the corresponding integral convolutions. A successive growth is shown for the orders of smallness of the residuals remaining after the substitution of the partial sums of the asymptotic solution into the equation under consideration. In addition, it is proved that there exists a unique classical solution and the constructed asymptotic series is its asymptotic expansion. The statement of the problem under consideration is also discussed in the light of known approaches to studying the Hamilton–Jacobi equation. The connection of the obtained result with the general theory of singularities of differentiable maps is shown.

Keywords: parabolic Hamilton–Jacobi type equation, simple boundary singularity, versal deformation, asymptotic solution, Laplace method.

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